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Heat transfer between a circular free impinging jet and a solid surface with non-uniform wall temperature or wall heat flux—1. Solution for the stagnation region

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Abstract—In this, the first part of a two-part analytical study of the heat transfer between an axisymmetrical free impinging jet and a solid flat surface with non-uniform wall temperature or wall heat flux, both the exact energy equation and the boundary layer energy equation are solved asymptotically in the vicinity of the stagnation point. The results show that the nonuniformity of wall temperature or wall heat flux has a considerable effect on the stagnation point Nusselt number. Increasing the wall temperature or wall heat flux with the radial distance reduces the stagnation point Nusselt number while decreasing the wall temperature or wall heat flux with r enhances the heat transfer at the stagnation point. For the special case of constant wall temperature or wall heat flux, the result is essentially the same as that of Sibulkin, and the difference in Nusselt number is less than 0.3% for $Pr \geqslant 0.7$.

1. INTRODUCTION

THE HEAT transfer between an impinging jet and a solid surface is of interest in many engineering applications, such as the annealing of metal and plastic sheets, the tempering of glass, the drying of textiles, veneer, paper, and the cooling of microelectronic equipment. As was pointed out in ref. [1], the heat transfer coefficients by jet impingement are several times greater than those by other flow patterns. Because of the high heat transfer rate associated with jet impingement, the technique is commonly used in practice. Although considerable experimental work on jet impingement heat transfer has been done, limited analytical studies have been carried out, especially for the case of a free impinging liquid jet and non-uniform wall temperature or wall heat flux. In contrast to a submerged jet, a free liquid jet is discharged into a gaseous rather than a liquid environment.

When a free liquid jet impinges on a flat solid surface, as shown in Fig. 1, the entire flow field is conventionally divided into two regions [2]—a potential region and a viscous region (boundary layer). The flow in the potential region may be treated as inviscid and the region may be subdivided into three subregions [3]—the free jet region, the impingement region and the wall jet region. The viscous region before the hydraulic jump may be subdivided into four subregions as follows [4].

(1) The stagnation region where the main flow velocity at the edge of the boundary layer grows rapidly from zero at the stagnation point to the jet velocity U_0. The radial dimension of the stagnation region is of the order of the jet radius r_0 and the boundary layer thickness is of the order of $\sqrt{(\nu r_0/U_0)}$, where ν is the kinematic viscosity of the fluid.

(2) The boundary layer region where the radial distance is greater than r_0. In this region, the velocity outside the boundary layer remains constant and the flow is of the boundary layer type.

(3) The transition region where the boundary layer becomes as thick as the whole liquid layer.

(4) The similarity region where the whole flow is of the boundary layer type and there exists a similarity solution to the velocity field.

In fact, there must be a transition region between the stagnation and the boundary layer regions because at $r \approx r_0$ the flow pattern is different from that in the neighborhood of the stagnation point and in the boundary layer region. In the present work, this region will be considered separately.

An experimental investigation of the hydrodynamic behavior of a single circular free liquid jet impinging on a horizontal surface was carried out by Olsson and Turkdogan [5], who measured the kinetic energy of the liquid film, the thickness of the liquid layer and the surface velocity of radial flow. Their measurements show that the surface velocity remains constant up to the hydraulic jump. This is different from the theoretical result of Watson [4] which shows that the surface velocity decreases as r increases in the similarity region.

Heat transfer near a stagnation point has been extensively investigated. Squire [6] and Sibulkin [7] were among the first to study stagnation point heat transfer. Squire obtained the exact solution to the

NOMENCLATURE

a constant in the solution of potential flow
c constant defined in equation $(38)_2$
$c_2, c_3, \ldots, c_n$ coefficients in prescribed wall temperature, equation (12)
$d_2, d_3, \ldots, d_n$ coefficients in prescribed wall heat flux, equation (13)
F_w dimensionless function defined in equation (9c), Q_w/Q_0
k conductivity
Nu Nusselt number, $2r_0Q_w/k(T_w-T_\infty)$
P pressure
Pr Prandtl number
Q_0 stagnation point heat flux
Q_w wall heat flux
r coordinate along the wall
r_0 jet radius
r_m radial distance of matching location
Re Reynolds number, $2U_0r_0/\nu$
T temperature
T_0 stagnation point temperature
T_m temperature at the matching location
T_w wall temperature
T_∞ jet temperature
u velocity in the r-direction
U_0 jet velocity
w velocity in the z-direction
z coordinate normal to the wall.

Greek symbols
α diffusivity
δ viscous boundary layer thickness
δ_s boundary layer thickness in stagnation region
δ_t thermal boundary layer thickness
Δ ratio of thermal boundary layer thickness to viscous boundary layer thickness
η dimensionless variable defined in equation (3c)
θ dimensionless temperature defined in equations (3a)
θ_m dimensionless temperature at the matching location
θ_w dimensionless temperature on the wall
λ_n eigenvalues given by $J_0(\lambda_n\xi_m) = 0$
ν kinematic viscosity
ξ dimensionless variable defined in equation (3b)
ξ_m dimensionless radius, $r_m/\sqrt{(\nu/a)}$
ρ density
ϕ dimensionless function defined in equations (5).

problem of steady, laminar flow heat transfer near a two-dimensional stagnation point with constant wall temperature. Sibulkin extended the problem to the case of the axisymmetrical stagnation point. Their solutions show that if the wall temperature is constant, the temperature distribution near the stagnation point is independent of the radial distance from the stagnation point. Transient heat transfer near a stagnation point has been examined by several investigators. Chao and Jeng [8] considered transient laminar forced convection heat transfer for a two-dimensional and axisymmetrical stagnation point due to time-dependent uniform wall temperature or wall heat flux. Sano [9] obtained an asymptotic solution for small Prandtl numbers to the problem of unsteady boundary layer in impulsive stagnation flow in which both the flow field and temperature field are unsteady. The effects of blowing and suction on transient heat transfer at a stagnation point due to a step change with time in wall temperature was analyzed by the same author [10]. Kumari and Nath [11] considered the combined effect of forced and free convection on unsteady laminar incompressible boundary layer flow with mass transfer at the stagnation point of a three-dimensional body with time-dependent wall temperature. More

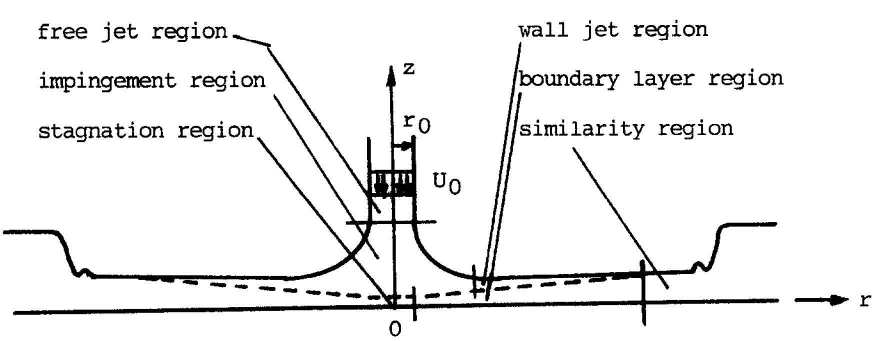

FIG. 1. Schematic diagram of a free impinging jet system.

recently, Smolsky *et al.* [12] presented experimental and theoretical results of heat transfer in the region near the stagnation point of a sphere and a transversely-oriented cylinder immersed in a gas flow under the unsteady conditions resulting from a stepwise change in the flow temperature or the body temperature.

In all previous solutions [6–12], the wall temperature or wall heat flux is either constant or a function of time only. Hence the temperature distribution near the stagnation point is independent of radial location. The boundary layer energy equation used in obtaining these solutions excludes the contribution of radial conduction. However, in many applications, wall temperature or wall heat flux may vary with the radial distance r. In such a case, the temperature distribution near the stagnation point will be a function of the axial distance z as well as r. Furthermore, if the variation of wall temperature or wall heat flux is sufficiently large such that the characteristic length for the temperature field in the r-direction is of the same order as the thermal boundary layer thickness, conduction in the radial direction cannot be neglected. Consequently, the boundary layer energy equation is not applicable and the exact energy equation should be used. Radial conduction plays a much more important role in the stagnation region than elsewhere since the velocity is so small and thus radial convection is negligible.

The conjugate heat transfer problem associated with jet impingement is especially important in engineering applications. The analytical solution presented in Parts 1 and 2 of the present study can be used to solve the corresponding conjugate problem in which a circular jet impinges on the top surface of a solid plate with prescribed temperature or heat flux at the bottom surface.

In Part 1 of the present study, the exact steady-state energy equation is solved asymptotically to examine the effect of wall temperature or wall heat flux on the heat transfer in the neighborhood of the stagnation point. The solution is therefore restricted to the vicinity of the stagnation point. It is found that heat transfer at the stagnation point can be significantly affected by wall temperature or wall heat flux distribution.

2. FORMULATION

For steady-state, incompressible flow with constant properties and negligible dissipation, the energy equation for axisymmetric flow is

$$u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}=\alpha\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right) \quad (1)$$

where T is the temperature, r and z the radial and axial coordinates measured from the stagnation point, u and w the velocity components along r and z, respectively, and α the thermal diffusivity. The velocity components u and w are obtained from the solution to axisymmetric flow in the stagnation region [13]. The boundary conditions are

$$\frac{\partial T}{\partial r}=0 \qquad \text{at } r=0 \quad (2a)$$

$$T=T_m(z) \qquad \text{at } r=r_m \quad (2b)$$

$$T=T_w(r) \text{ or } Q=Q_w(r) \qquad \text{at } z=0 \quad (2c)$$

$$T=T_\infty=\text{constant} \qquad \text{as } z\to\infty \quad (2d)$$

where T_∞ is the jet temperature, $T_w(r)$ the wall temperature, and $Q_w(r)$ the wall heat flux. $T_m(z)$ is the temperature distribution at $r=r_m$ which is determined by matching the solution in the stagnation region with the solution outside the stagnation region. The matching surface $r=r_m$ is the location where the boundary layer thickness in the stagnation region is matched with that outside the stagnation region.

To nondimensionalize the problem the following dimensionless variables are introduced:

$$\theta=\begin{cases}(T-T_\infty)/(T_0-T_\infty) & \text{for prescribed wall temperature}\\ k(T-T_\infty)/(Q_0\sqrt{(\nu/a)}) & \text{for prescribed wall heat flux}\end{cases} \quad (3a)$$

$$\xi=r/\sqrt{(\nu/a)} \quad (3b)$$

$$\eta=z/\sqrt{(\nu/a)} \quad (3c)$$

where k is the thermal conductivity, ν the kinematic viscosity, T_0 and Q_0 the temperature and heat flux at the stagnation point, respectively, and a is a constant. The similarity solution to the flow field in the stagnation region gives [13]

$$u=\sqrt{(a\nu)}\xi\phi'(\eta) \quad (4a)$$

$$w=-2\sqrt{(a\nu)}\phi(\eta) \quad (4b)$$

where $\phi(\eta)$ is a transformation function defined by

$$\phi'''+2\phi\phi''-\phi'^2+1=0 \quad (5a)$$

$$\phi(0)=\phi'(0)=0,\quad \phi'(\infty)=1. \quad (5b)$$

The viscous boundary layer thickness δ_s is given by

$$\delta_s=1.98\sqrt{(\nu/a)}. \quad (6)$$

For the case of a circular impinging jet, constant a is given by [2]

$$a=0.44\frac{U_0}{r_0} \quad (7)$$

where U_0 is the jet velocity and r_0 the jet radius. Using definitions (3) and flow field solution (4), the dimensionless form of the stagnation region energy equation and boundary conditions become

$$\xi\phi'\frac{\partial\theta}{\partial\xi}-2\phi\frac{\partial\theta}{\partial\eta}=\frac{1}{Pr}\left(\frac{\partial^2\theta}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial\theta}{\partial\xi}+\frac{\partial^2\theta}{\partial\eta^2}\right) \quad (8)$$

and

$$\frac{\partial \theta}{\partial \xi} = 0 \quad \text{at } \xi = 0 \tag{9a}$$

$$\theta = \theta_m(\eta) \quad \text{at } \xi = \xi_m \tag{9b}$$

$$\theta = \theta_w(\xi) \text{ or } -\frac{\partial \theta}{\partial \eta} = \frac{Q_w}{Q_0} = F_w(\xi) \text{ at } \eta = 0 \tag{9c}$$

$$\theta = 0 \quad \text{at } \eta = \infty \tag{9d}$$

where Pr is the Prandtl number, $\theta_m(\eta)$ the dimensionless temperature at the dimensionless radius ξ_m, and

$$\theta_w = (T_w - T_\infty)/(T_0 - T_\infty).$$

3. SMALL VARIATION OF WALL TEMPERATURE OR WALL HEAT FLUX

In the case where the variation of wall temperature or wall heat flux is sufficiently small so that the characteristic length of the temperature field in the r-direction is much larger than the thermal boundary layer thickness, conduction in the r-direction can be neglected. Hence energy equation (8) and boundary conditions (9) reduce to

$$\frac{\partial^2 \theta}{\partial \eta^2} + 2Pr\,\phi\frac{\partial \theta}{\partial \eta} - Pr\,\xi\phi'\frac{\partial \theta}{\partial \xi} = 0 \tag{10}$$

and

$$\frac{\partial \theta}{\partial \xi} = 0 \quad \text{at } \xi = 0 \tag{11a}$$

$$\theta = \theta_w(\xi) \text{ or } -\frac{\partial \theta}{\partial \eta} = F_w(\xi) \quad \text{at } \eta = 0 \tag{11b}$$

$$\theta = 0 \quad \text{at } \eta = \infty. \tag{11c}$$

The prescribed wall temperature $\theta_w(\xi)$ or wall heat flux $Q_w(\xi)$ may be any continuous function of ξ, which satisfies the symmetry condition $d\theta_w(0)/d\xi = 0$ or $dQ_w(0)/d\xi = 0$. Corresponding to the definition in equation (3), $\theta_w(\xi)$ may be expanded in a Taylor series near $\xi = 0$ as

$$\theta_w(\xi) = 1 + c_2\xi^2 + c_3\xi^3 + \ldots + c_n\xi^n + \ldots \tag{12}$$

where c_n are constants. Similarly, the wall heat flux may be written as

$$-\frac{\partial \theta(\xi, 0)}{\partial \eta} = F_w(\xi) = 1 + d_2\xi^2 + d_3\xi^3 + \ldots + d_n\xi^n + \ldots \tag{13}$$

where d_n are constants. Since the solution is sought in the vicinity of the stagnation point, the dependent variable θ can be expanded in a Taylor series as follows:

$$\theta = A_0(\eta) + A_1(\eta)\xi + A_2(\eta)\xi^2 + A_3(\eta)\xi^3 + \ldots + A_n(\eta)\xi^n + \ldots \tag{14}$$

Because of the symmetry, the radial heat flux at $r = 0$ must be zero, hence $A_1(\eta)$ in equation (14) should vanish. Substituting equation (14) into equation (10), the governing ordinary differential equations for coefficients A_n are obtained

$$A_n'' + 2Pr\,\phi A_n' - n\,Pr\,\phi' A_n = 0, \quad n = 0, 2, 3, 4\ldots \tag{15}$$

The corresponding boundary conditions for the prescribed wall temperature are

$$A_0(0) = 1, \quad A_n(0) = c_n, \quad n = 2, 3, 4\ldots \tag{16a}$$

$$A_n(\infty) = 0, \quad n = 0, 2, 3, 4\ldots \tag{16b}$$

For prescribed wall heat flux the boundary conditions are

$$A_0'(0) = -1, \quad A_n'(0) = -d_n, \quad n = 2, 3, 4\ldots \tag{17a}$$

$$A_n(\infty) = 0, \quad n = 0, 2, 3, 4\ldots \tag{17b}$$

3.1. *Solution for non-uniform wall temperature*

The solution of $A_0(\eta)$ with boundary conditions (16) can be obtained in closed form

$$A_0(\eta) = 1 - \frac{\int_0^\eta \exp\left[-2Pr\int_0^\eta \phi\,d\eta\right]d\eta}{\int_0^\infty \exp\left[-2Pr\int_0^\eta \phi\,d\eta\right]d\eta} \tag{18}$$

while the remaining $A_n(\eta)$ coefficients may be solved numerically. Once $A_n(\eta)$ are obtained, the temperature field is determined by equation (14) and the wall heat flux can be obtained as follows:

$$q_w = -k\frac{\partial T(r, 0)}{\partial z} = -k(T_0 - T_\infty)\sqrt{(a/\nu)}[A_0'(0) + A_2'(0)\xi^2 + \ldots + A_n'(0)\xi^n + \ldots]. \tag{19}$$

Thus the local Nusselt number is given by

$$Nu = -0.9381Re^{1/2} \times \frac{A_0'(0) + A_2'(0)\xi^2 + A_3'(0)\xi^3 + \ldots + A_n'(0)\xi^n + \ldots}{1 + c_2\xi^2 + c_3\xi^3 + \ldots + c_n\xi^n + \ldots} \tag{20}$$

where Re is the Reynolds number defined as $2U_0r_0/\nu$.

3.2. *Solution for non-uniform wall heat flux*

The solution of $A_0(\eta)$ with boundary conditions (17) is

$$A_0(\eta) = \int_0^\infty \exp\left[-2Pr\int_0^\eta \phi\,d\eta\right]d\eta - \int_0^\eta \exp\left[-2Pr\int_0^\eta \phi\,d\eta\right]d\eta. \tag{21}$$

The solutions to $A_n(\eta)$, $n = 2, 3, 4, \ldots$ with boundary conditions (17) may be obtained numerically. Once $A_n(\eta)$ are determined, the wall temperature is given by

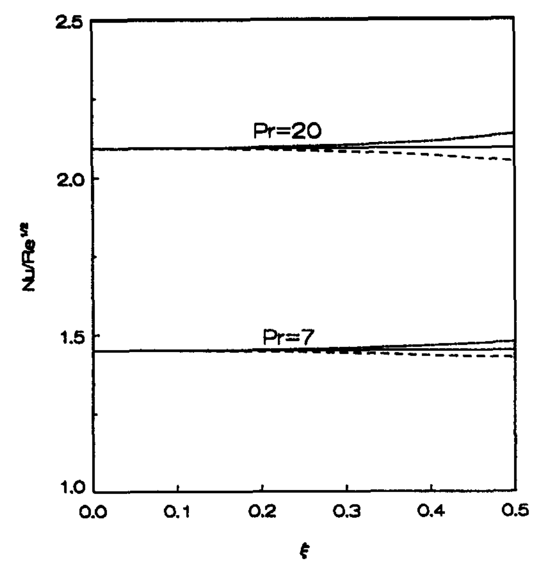

FIG. 2. Nusselt number profile for non-uniform wall temperature with radial conduction neglected: —·—, $C_2 = C_3 = C_4 = 0.1$, increasing wall temperature; ——, $C_2 = C_3 = C_4 = 0$, constant wall temperature; ---, $C_2 = C_3 = C_4 = -0.1$, decreasing wall temperature.

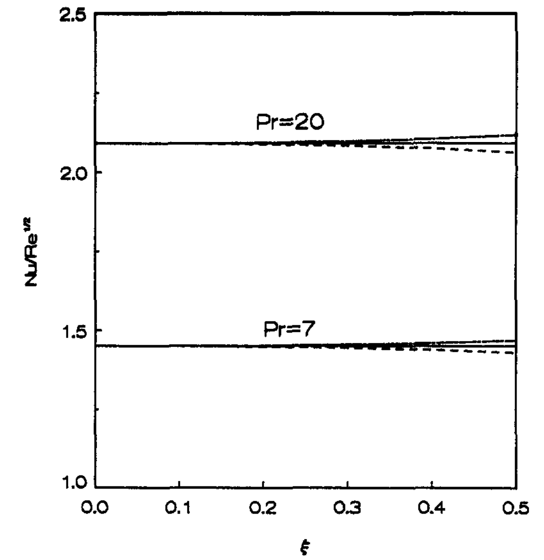

FIG. 3. Nusselt number profile for non-uniform wall heat flux with radial conduction neglected: —·—, $d_2 = d_3 = d_4 = 0.1$, increasing wall heat flux; ——, $d_2 = d_3 = d_4 = 0$, constant wall heat flux; ---, $d_2 = d_3 = d_4 = -0.1$, decreasing wall heat flux.

$$T_w = T_\infty + \frac{Q_0\sqrt{(\nu/a)}}{k}[A_0(0)+A_2(0)\xi^2+A_3(0)\xi^3 +\ldots+A_n(0)\xi^n+\ldots]. \quad (22)$$

The local Nusselt number is obtained in the form

$$Nu = 0.9381Re^{1/2} \times \frac{1+d_2\xi^2+d_3\xi^3+\ldots+d_n\xi^n+\ldots}{A_0(0)+A_2(0)\xi^2+A_3(0)\xi^3+\ldots+A_n(0)\xi^n+\ldots}. \quad (23)$$

For the special case of constant wall temperature or wall heat flux, i.e. $c_2 = c_3 = \ldots = c_n = 0$ or $d_2 = d_3 = \ldots = d_n = 0$, equations (20) and (23) reduce to

$$Nu = \frac{0.9381Re^{1/2}}{\int_0^\infty \exp\left[-2Pr\int_0^\eta \phi\,\mathrm{d}\eta\right]\mathrm{d}\eta} \quad (24)$$

which is identical to the result of Sibulkin [7].

The Nusselt number profiles for both cases are shown graphically in Figs. 2 and 3. It is clear from equations (18), (20), (21) and (23) that the wall temperature or wall heat flux distribution does not affect the stagnation point Nusselt number. The reason for this is that the boundary layer energy equation is used in obtaining the solution in which radial conduction is neglected. As shown in the following sections, the stagnation point Nusselt number can be considerably affected by the distribution of wall temperature or wall heat flux when radial conduction is taken into consideration.

4. SOLUTION FOR STEEP VARIATION OF WALL TEMPERATURE OR WALL HEAT FLUX

The solution in the previous section is valid for the case where the variation of wall temperature or wall heat flux is small so that the boundary layer energy equation is applicable. If, however, the wall temperature or wall heat flux varies steeply with r, conduction in the radial direction must be included. Hence the complete energy equation should be solved. Nevertheless, one can show that the first term in equation (8) which represents radial convection can be neglected in the vicinity of the stagnation point. Substitution of equation (14) into equation (8) shows that the first term in equation (8) is of the order of ξ^2 and all other terms are of the order of 1. Hence, in the vicinity of the stagnation point, i.e. for small ξ, equation (8) simplifies to

$$\frac{\partial^2\theta}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial\theta}{\partial\xi}+\frac{\partial^2\theta}{\partial\eta^2}+2Pr\,\phi\frac{\partial\theta}{\partial\eta} = 0. \quad (25)$$

Since ϕ is independent of ξ, equation (25) can be solved by the method of separation of variables. The solution is

$$\theta = \frac{\xi^2}{\xi_m^2}\theta_m(\eta) + \sum_{n=1}^{\infty}[g_n(\eta)+D_n(\eta)]J_0(\lambda_n\xi) \quad (26)$$

where g_n is governed by the following ordinary differential equations:

$$g_n''+2Pr\,\phi g_n'-\lambda_n^2 g_n = 0 \quad (27a)$$

$$g_n(0) = \frac{2}{G}\int_0^{\xi_m}\left[\theta_w(\xi) - \frac{\xi^2}{\xi_m^2}\theta_m(0)\right]J_0(\lambda_n\xi)\xi\,d\xi$$

for prescribed T_w (27b)

$$g_n'(0) = -\frac{2}{G}\int_0^{\xi_m}\left[F_w(\xi) + \frac{\xi^2}{\xi_m^2}\theta_m'(0)\right]J_0(\lambda_n\xi)\xi\,d\xi$$

for prescribed Q_w (27c)

$$g_n(\infty) = 0 \quad (27d)$$

and D_n is defined by the following equations:

$$D_n'' + 2Pr\,\phi D_n' - \lambda_n^2 D_n = L(\eta) \quad (28a)$$

$$D_n(0) = 0 \quad \text{for prescribed } T_w \quad (28b)$$

$$D_n'(0) = 0 \quad \text{for prescribed } Q_w \quad (28c)$$

$$D_n(\infty) = 0 \quad (28d)$$

where

$$L(\eta) = -\frac{8\theta_m}{G\lambda_n\xi_m}J_1(\lambda_n\xi_m) - \frac{2}{G\lambda_n^2}[\lambda_n\xi_m J_1(\lambda_n\xi_m) - 2J_2(\lambda_n\xi_m)](\theta_m'' + 2Pr\,\phi\theta_m') \quad (29)$$

and λ_n is given by $J_0(\lambda_n\xi_m) = 0$. G in the above equations is given as follows:

$$G = \xi_m^2[J_1(\lambda_n\xi_m)]^2. \quad (30)$$

Equations (27) can be solved numerically for any prescribed $\theta_w(\xi)$ or $F_w(\xi)$ since $\theta_m(0) = \theta_w(\xi_m)$ and $\theta_m'(0) = -F_w(\xi_m)$ in the boundary conditions. In order to solve equations (28), it is necessary to know $\theta_m(\eta)$ which should be determined by matching this solution with the solution outside the stagnation region. Before proceeding further, the solution in the transition region between the stagnation region and the boundary layer region will be obtained and $\theta_m(\eta)$ will be determined.

5. INTEGRAL SOLUTION IN THE TRANSITION REGION

The transition region through which the flow field changes with r from the stagnation type to the boundary layer type is located at $r \approx r_0$ where the radial velocity is well developed and radial convection is much more important than radial conduction. Hence the latter can be neglected and the temperature field is of the boundary layer type. The integral form of the momentum and energy equations for radial flow is

$$\frac{d}{dr}\left[\int_0^{\delta} ru(U_0 - u)\,dz\right] = vr\frac{\partial u(r,0)}{\partial z} \quad (31)$$

and

$$\frac{d}{dr}\left[\int_0^{\delta_t} ru(T - T_\infty)\,dz\right] = -\frac{vr}{Pr}\frac{\partial T(r,0)}{\partial z}. \quad (32)$$

The velocity and temperature distributions may be assumed to have the forms

$$u = U_0\left(\frac{2z}{\delta} - \frac{2z^3}{\delta^3} + \frac{z^4}{\delta^4}\right) \quad (33)$$

and

$$T - T_\infty = [T_w(r) - T_\infty]\left(1 - \frac{2z}{\delta_t} + \frac{2z^3}{\delta_t^3} - \frac{z^4}{\delta_t^4}\right) \quad (34)$$

for prescribed wall temperature or

$$T - T_\infty = \frac{Q_w(r)\delta_t}{k}\left(\frac{1}{2} - \frac{z}{\delta_t} + \frac{z^3}{\delta_t^3} - \frac{z^4}{2\delta_t^4}\right) \quad (35)$$

for prescribed wall heat flux. δ and δ_t are the viscous and thermal boundary layer thicknesses, respectively. Introduction of equation (33) into equation (31) yields

$$\delta(r) = \sqrt{\left(\frac{420}{37}\frac{v}{U_0}r + \frac{c}{r^2}\right)} \quad (36)$$

where c is a constant to be determined by matching equation (36) with equation (6) at $r = r_m$. The matching conditions are

$$\delta_s(r_m) = \delta(r_m), \quad \frac{d}{dr}\delta_s(r_m) = \frac{d}{dr}\delta(r_m) \quad (37)$$

which give

$$r_m = 0.52r_0, \quad c = 0.81\frac{v}{U_0}r_0^3. \quad (38)$$

Substituting equations (33) and (34) into equation (32), one obtains an ordinary differential equation which determines Δ, the ratio of thermal boundary layer thickness to the viscous boundary layer thickness for the case of prescribed wall temperature

$$\frac{d}{dr}[r\delta(r)(T_w - T_\infty)H(\Delta)] = \frac{2vr}{U_0 Pr}\frac{T_w(r) - T_\infty}{\delta(r)\Delta} \quad (39)$$

where

$$H(\Delta) = \begin{cases} \frac{2}{15}\Delta^2 - \frac{3}{140}\Delta^4 + \frac{1}{180}\Delta^5 & Pr \geq 1 \\ \frac{3}{10}\Delta - \frac{3}{10} + \frac{2}{15\Delta} - \frac{3}{140\Delta^3} + \frac{1}{180\Delta^4} & Pr < 1 \end{cases} \quad (40)$$

and Δ is given by $\Delta = \delta_t/\delta$. For the special case of constant wall temperature, equation (39) is identical to that given in ref. [14]. For the case of prescribed wall heat flux, substitution of equations (33) and (35) into equation (32) leads to

$$\frac{d}{dr}[rQ_w(r)\delta^2(r)\Delta H(\Delta)] = \frac{2v}{Pr\,U_0}rQ_w(r). \quad (41)$$

The boundary condition $\Delta(\xi_m) = \Delta_m$ needed to solve equations (39) and (41) is determined in such a way that the curve of the Nusselt number profile is smooth at $\xi = \xi_m$. Computations show that the Nusselt number near the stagnation point is not sensitive to Δ_m.

For $\Delta_m = 0.5/Pr^{1/3}$ and $3/Pr^{1/3}$, for instance, the difference in the stagnation point Nusselt number is less than 1%. Once Δ is obtained, the temperature field in the transition region is known and $\theta_m(\eta) = \theta(\xi_m, \eta)$ can be determined.

6. RESULTS AND DISCUSSION

Once equations (27) and (28) are solved, the temperature distribution in the neighborhood of the stagnation point is given by equation (26), hence the heat transfer coefficient can be determined.

6.1. *Result for prescribed wall temperature*

The heat flux on the wall is

$$q_w = -k\frac{T_0 - T_\infty}{\sqrt{(\nu/a)}}\left\{\frac{\xi^2}{\xi_m^2}\theta'_m(0) + \sum_{n=1}^{\infty}[g'_n(0) + D'_n(0)]J_0(\lambda_n\xi)\right\} \quad (42)$$

and the Nusselt number is given by

$$Nu = -0.9381\frac{Re^{1/2}}{\theta_w}\left\{\frac{\xi^2}{\xi_m^2}\theta'_m(0) + \sum_{n=1}^{\infty}[g'_n(0) + D'_n(0)]J_0(\lambda_n\xi)\right\}. \quad (43)$$

6.2. *Result for prescribed wall heat flux*

The wall temperature is

$$T_w - T_\infty = \frac{Q_0\sqrt{(\nu/a)}}{k}\left\{\frac{\xi^2}{\xi_m^2}\theta_m(0) + \sum_{n=1}^{\infty}[g_n(0) + D_n(0)]J_0(\lambda_n\xi)\right\} \quad (44)$$

thus the Nusselt number is

$$Nu = \frac{0.9381 Re^{1/2}F_w(\xi)}{\frac{\xi^2}{\xi_m^2}\theta_m(0) + \sum_{n=1}^{\infty}[g_n(0) + D_n(0)]J_0(\lambda_n\xi)}. \quad (45)$$

In order to examine the effect of wall temperature or wall heat flux on the heat transfer near the stagnation point, the Nusselt number has been calculated for different profiles of prescribed wall temperature and wall heat flux as well as different Prandtl numbers. Figure 4 shows these wall temperature and wall heat flux profiles graphically. Curve 1 represents decreasing wall temperature or wall heat flux with r and curve 2 is for constant wall temperature or wall heat flux. Curves 3 and 4 represent slowly and steeply increasing wall temperature or wall heat flux, respectively. Computations show that for the special case of constant wall temperature or wall heat flux, the Nusselt number is essentially the same as that of Sibulkin [7], the difference being less than 0.3% for $Pr \geq 0.7$. Radial variation of Nusselt number is shown in Figs. 5 and 6 for $Pr = 0.7$, 7, and 20 corresponding to air, water and fluorocarbon liquids, respectively.

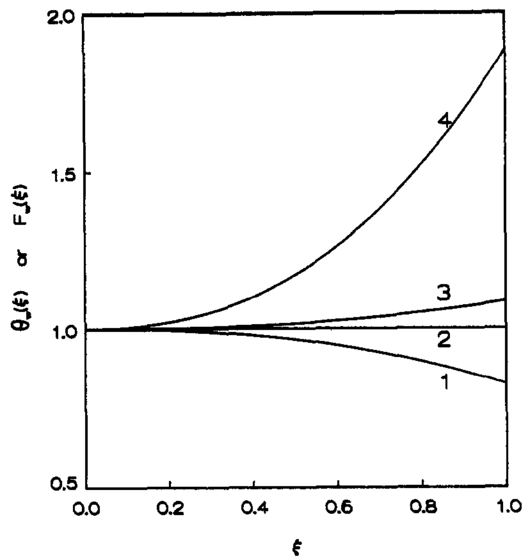

FIG. 4. Prescribed wall temperature or wall heat flux: 1, decreasing wall temperature or wall heat flux; 2, constant wall temperature or wall heat flux; 3, slowly increasing wall temperature or wall heat flux; 4, steeply increasing wall temperature or wall heat flux.

It can be seen from Figs. 5 and 6 that the stagnation point heat transfer can be influenced considerably by the wall temperature or wall heat flux distribution. The reason for this is that when the wall temperature or wall heat flux varies appreciably with r, the characteristic length of the temperature field in the r-direction may be of the same order as that in the z-direction. In such a case, conductions in the r-direction

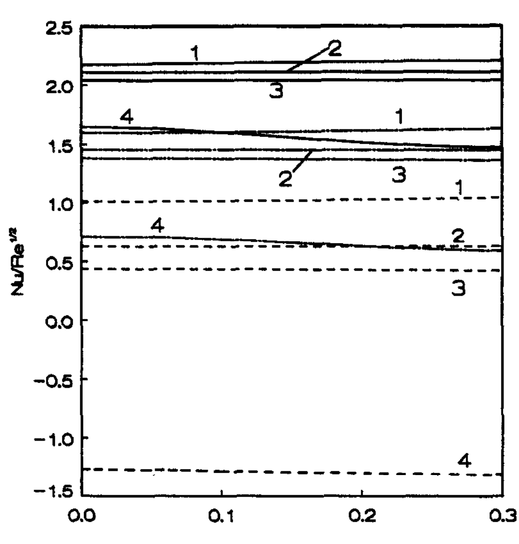

FIG. 5. Nusselt number profile for non-uniform wall temperature shown in Fig. 4: 1, for decreasing wall temperature; 2, for constant wall temperature; 3, for slowly increasing wall temperature; 4, for steeply increasing wall temperature. ——, $Pr = 20$; —·—, $Pr = 7$; ---, $Pr = 0.7$.

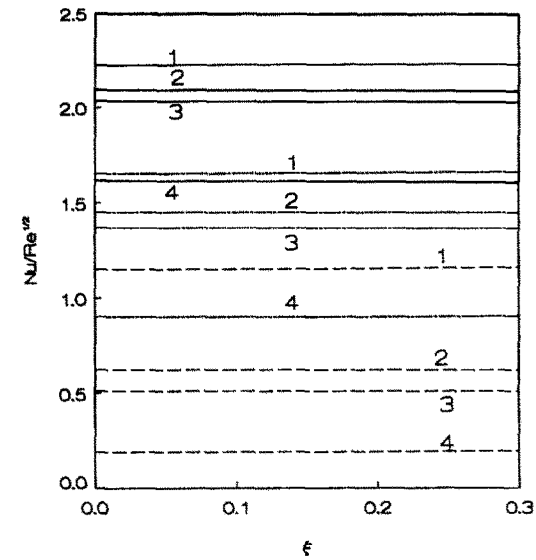

FIG. 6. Nusselt number profile for non-uniform wall heat flux shown in Fig. 4: 1, for decreasing wall heat flux; 2, for constant wall heat flux; 3, for slowly increasing wall heat flux; 4, for steeply increasing wall heat flux. ——, $Pr = 20$; —·—, $Pr = 7$; ---, $Pr = 0.7$.

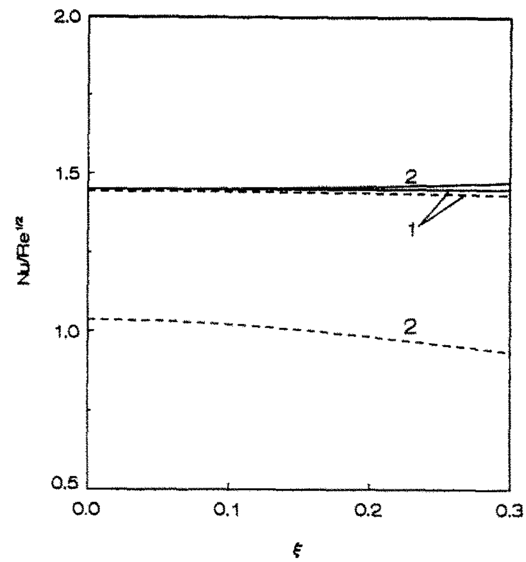

FIG. 7. Comparison between the results with and without radial conduction, $Pr = 7$: 1, $C_2 = C_3 = C_4 = 0.01$, very slowly increasing wall temperature; 2, $C_2 = C_3 = C_4 = 0.3$, steeply increasing wall temperature. ---, with radial conduction; ——, without radial conduction.

and in the z-direction are comparable. Furthermore, near the stagnation point the velocity is small so that radial convection is negligible as shown in Section 4. Hence, conduction dominates in the r-direction near the stagnation point. The fact that the velocity is small in the vicinity of the stagnation point can be seen in equations (4) and (5) which show that u is of the order of $\xi\eta$ and w is of the order of η^2.

As shown in Figs. 5 and 6, increasing the wall temperature or wall heat flux with r reduces the stagnation point Nusselt number. For slowly increasing wall temperature or wall heat flux, the reduction is small. For steeply increasing wall temperature or wall heat flux, however, the stagnation point Nusselt number is considerably reduced. For the wall temperature and wall heat flux distribution represented by curve 4 in Fig. 4, for instance, the reduction in the Nusselt number for $Pr = 7$ is approximately 50 and 40%, respectively. When the wall temperature or wall heat flux decreases as r increases, stagnation point Nusselt number is higher than that for constant wall temperature or wall heat flux. This is due to the fact that heat is conducted radially from the stagnation point when the wall temperature or wall heat flux decreases with r. Consequently, the heat flux from the wall to the fluid at the stagnation point is increased. On the other hand, when the wall temperature or wall heat flux increases with r, heat is conducted towards the stagnation point and consequently, the stagnation point Nusselt number is reduced. The figures also indicate that the effect of wall temperature or wall heat flux on the stagnation point Nusselt number becomes more pronounced as the Prandtl number is decreased. This is due to the fact that the radial conduction plays a more important role when the Prandtl number is smaller. It is worth noting that the stagnation point Nusselt number can be negative when the wall temperature increases with r very steeply. In such a case, the effect of radial conduction is so severe that the fluid becomes hotter than the wall near the stagnation point and, hence, heat flows from the fluid to the wall although the wall is hotter than the jet.

In Fig. 7, the Nusselt number is compared with the result obtained in Section 3 in which the radial conduction is neglected for the case of non-uniform wall temperature. The comparison is given for two different wall temperature profiles. The first one is for the case where the characteristic length of the temperature field in the r-direction is much smaller than that in the z-direction, while the second one is for the case where the two characteristic lengths are comparable. It is clear from Fig. 7 that for the first case the two curves are essentially the same. It follows that radial conduction can be neglected in such a case. For the second case, however, the difference is about 35% and hence radial conduction cannot be ignored.

The sensitivity of the solution to the value of r_m, the radial distance from the stagnation point to the matching point, is investigated by calculating the Nusselt number for $\xi_m = 15$ and 25 for a typical case of moderately increasing wall temperature and $Pr = 7$. The result shows that the Nusselt number near the stagnation point is essentially the same for $\xi_m = 15$ and 25. Hence the solution is not sensitive to the value of r_m.

In order to examine the error of the solution resulting from neglecting the radial convection term near the stagnation point, we consider a simplified case where radial conduction is neglected. In such a case, the solution to the energy equation can be obtained analytically for the cases with and without the radial convection term. The difference between the two solutions can be taken as an upper bound of the error resulting from neglecting the radial convection term in our solution since the radial convection plays a less important role when radial conduction is included than when it is neglected. For the first case where the radial convection is included, the solution was obtained in Section 3. For the second case where both radial convection and conduction are neglected, the energy equation (1) becomes

$$\alpha\frac{\partial^2 T}{\partial z^2} - w\frac{\partial T}{\partial z} = 0 \tag{46}$$

and the boundary conditions are

$$\frac{\partial T}{\partial r} = 0 \quad \text{at } r = 0 \tag{47a}$$

$$T = T_w(r) \quad \text{at } z = 0 \tag{47b}$$

$$T = T_\infty \quad \text{at } z = \infty. \tag{47c}$$

The solution to equation (46) with boundary conditions (47) can be obtained by the similarity method. The result is

$$\frac{T - T_\infty}{T_w - T_\infty} = 1 - \frac{\int_0^\eta \exp\left[-2Pr\int_0^\eta \phi\,d\eta\right]d\eta}{\int_0^\infty \exp\left[-2Pr\int_0^\eta \phi\,d\eta\right]d\eta}. \tag{48}$$

The wall heat flux can be obtained as

$$q_w = \frac{k(T_w - T_\infty)}{\sqrt{(\nu/a)}\int_0^\infty \exp\left[-2Pr\int_0^\eta \phi\,d\eta\right]d\eta}. \tag{49}$$

Thus the Nusselt number is

$$Nu = \frac{0.9381Re^{1/2}}{\int_0^\infty \exp\left[-2Pr\int_0^\eta \phi\,d\eta\right]d\eta} \tag{50}$$

which is independent of the wall temperature profile and is the same as equation (24). This result is plotted in Fig. 8 and compared with the result including radial convection. The figure shows that the two results are exactly the same at $\xi = 0$. This is because the velocity is zero at $\xi = 0$ and hence the effect of radial convection vanishes. For $\xi > 0$, however, the two results are slightly different and the difference increases with r. When the wall temperature or wall heat flux increases with r, the Nusselt number with the effect of radial convection is slightly higher than that without the effect of radial convection. The reason is that the radial convection enhances the wall heat flux when the wall temperature increases with r and reduces the wall temperature when the wall heat flux increases with r. On the other hand, the Nusselt number including the radial convection is slightly lower than that excluding the radial convection if the wall temperature

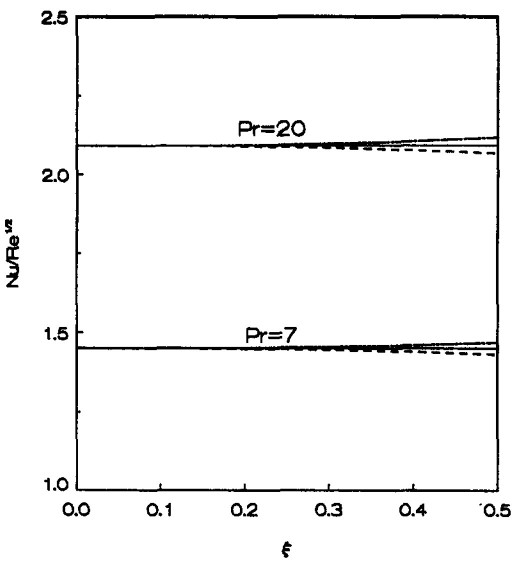

FIG. 8. Comparison between the results with and without radial convection: —·—, with radial convection, increasing wall temperature; ——, without radial convection, arbitrary wall temperature; – – –, with radial convection, decreasing wall temperature.

or wall heat flux decreases as r increases. This is due to the fact that the radial convection reduces the wall heat flux if the wall temperature decreases with r and increases the wall temperature if the wall heat flux decreases with r. For $\xi \ll 1$, the difference is negligibly small hence neglecting the radial convection is a reasonable simplification near the stagnation point.

7. CONCLUSIONS

The heat transfer near the stagnation point of a circular free impinging jet with non-uniform wall temperature or wall heat flux has been investigated analytically. The solution shows the considerable effect of wall temperature or wall heat flux distribution on the stagnation point Nusselt number when the variation of wall temperature or wall heat flux is not small. The reason is that the radial velocity is so small that the radial convection is negligible in the vicinity of the stagnation point. Hence radial conduction plays a much more important role near the stagnation point than it does in other regions. Consequently, considerable error would result from using the boundary layer energy equation in the presence of an appreciable variation of wall temperature or wall heat flux.

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TRANSFERT THERMIQUE ENTRE UN JET CIRCULAIRE, LIBRE, INCIDENT ET UNE SURFACE SOLIDE AVEC TEMPERATURE PARIETALE OU FLUX DE CHALEUR PARIETAL NON UNIFORME—1. SOLUTION POUR LA REGION D'ARRET

Résumé—Dans la première partie de cette étude analytique du transfert thermique entre un jet libre, axisymétrique et une surface solide à température ou flux de chaleur non uniforme sur la paroi, l'équation exacte de l'énergie et l'équation d'énergie pour la couche limite sont résolues asymptotiquement au voisinage du point d'arrêt. Les résultats montrent que la non uniformité de la température pariétale ou du flux de chaleur a une influence considérable sur le nombre de Nusselt au point d'arrêt. Un accroissement de la température ou du flux à la paroi avec la distance radiale r réduit le nombre de Nusselt tandis que la décroissance des mêmes paramètres augmente le transfert de chaleur au point d'arrêt. Dans le cas particulier d'une température ou d'un flux constant, le résultat est identique à celui de Sibulkin, et la différence en nombre de Nusselt est inférieure à 0,3% pour $Pr \geqslant 0,7$.

WÄRMEÜBERGANG ZWISCHEN EINEM KREISFÖRMIGEN FREI AUFTREFFENDEN STRAHL UND EINER FESTEN OBERFLÄCHE MIT UNGLEICHFÖRMIGER WANDTEMPERATUR ODER WÄRMESTROMDICHTE—1. LÖSUNGEN FÜR DEN STAGNATIONSBEREICH

Zusammenfassung—Es wird die Wärmeübertragung zwischen einem achsensymmetrischen frei auftreffenden Strahl und einer festen ebenen Oberfläche mit ungleichförmiger Wandtemperatur oder Wärmestromdichte analytisch untersucht. Es wird die Energieleichung im vollständiger Form und mit Grenzschichtvereinfachungen asymptotisch in der Nähe des Stagnationspunktes gelöst. Die Ergebnisse zeigen, daß die Ungleichförmigkeit der Wandtemperatur oder Wärmestromdichte einen nennenswerten Einfluß auf die Nusseltzahl im Stagnationspunkt hat. Nimmt die Wandtemperatur oder die Wärmestromdichte mit dem radialen Abstand vom Stagnationspunkt zu, so wird die Nusseltzahl kleiner—und umgekehrt. Für den Spezialfall konstanter Wandtemperatur oder Wärmestromdichte ist das Ergebnis im wesentlichen gleich dem von Sibulkin. Die Unterschiede in den Nusseltzahlen sind für $Pr \geqslant 0,7$ kleiner als 0,3%.

ТЕПЛОПЕРЕНОС МЕЖДУ КРУГЛОЙ СВОБОДНО ПАДАЮЩЕЙ СТРУЕЙ И ТВЕРДОЙ ПОВЕРХНОСТЬЮ С НЕОДНОРОДНЫМ РАСПРЕДЕЛЕНИЕМ ТЕМПЕРАТУРЫ ИЛИ ТЕПЛОВОГО ПОТОКА—1. РЕШЕНИЕ ДЛЯ ОБЛАСТИ ТОРМОЖЕНИЯ

Аннотация—В первой части аналитического исследования теплопереноса между осесимметричной свободно падающей струей и твердой плоской поверхностью с неоднородным распределением температуры или теплового потока асимптотически решаются общее уравнение энергии и уравнение энергии для пограничного слоя вблизи точки торможения. Результаты показывают, что неоднородность температуры или теплового потока на стенке оказывает значительное влияние на число Нуссельта в точке торможения. При увеличении температуры или теплового потока на стенке в радиальном направлении уменьшается число Нуссельта в точке торможения, уменьшение же температуры или теплового потока на стенке с r приведет к усилению теплопреноса в точке торможения. В частном случае постоянных температуры или теплового потока на стенке полученные результаты совпадают с данными Сибулкина и различие в числах Нуссельта не превосходит 0,3% для $Pr \geqslant 0,7$.